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The Cluster Expansion for Classical and Quantum Lattice Systems

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We develop a high-temperature expansion for general lattice systems which can be applied to classical as well as quantum systems. Applying the expansion we prove analyticity of correlation functions, uniqueness of equilibrium states, and cluster properties for classical and quantum lattice systems in the high-temperature region.

KEY WORDS: The cluster expansion; lattice systems; correlation functions; equilibrium states; KMS states; Araki's Gibbs condition; the method of integral equations.

1. INTRODUCTION

Our purpose here is to develop a high-temperature expansion method for statistical mechanical systems on lattice spaces which can be applied to classical as well as quantum systems. There are many previous studies of classical lattice systems(2,3,5,7,9,10,14,16) and quantum lattice systems(6,11) at high temperature and small activity by means of expansion methods, and various properties such as analyticity of the pressure and the correlation functions, clustering, and uniqueness of equilibrium states were well established for a large class of models. But the known methods(2,5,14) for classical lattice systems cannot be applied directly to the quantum lattice systems. Also it seems not obvious that the expansion method for quantum systems(6,11) can be applied to general classical lattice systems. (In principle, it may be applicable to discrete and bounded classical spin systems.)

Recently we came to know that Fröhlich(4) has found a cluster expansion which can be applied to classical as well as quantum lattice

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systems under the condition that interactions are of finite range. He asked us whether it is possible to extend the method to a large class of interactions which are not necessarily of finite range. The high-temperature expansion we develop is in a sense a modification of Fröhlich's method which can be applied to a general class of interactions. Our expansion method does not depend at all on whether the systems are classical or quantum. In this sense the expansion method can be viewed as the canonical expansion for general lattice systems. Applying our method we prove the analyticity of correlation functions with respect to temperature, uniqueness of equilibrium states, and cluster properties in the region of high temperatures.

In this paper we confine ourselves to a high-temperature expansion for bounded lattice systems. However, the method can be extended to a class of unbounded classical spin systems as in Ref. 10. Also, with a slight modification of the method one may easily develop a small activity expansion for lattice gases. To make the main idea clear and to avoid additional notational complications, we will not consider those cases in this paper.

Let us recall what we mean by a general lattice system.[8,14] We first consider quantum systems and then we describe classical systems. Let $\Lambda \subset Z^\nu$ be a bounded subset in the ν-dimensional lattice space Z^ν and let $\mathcal{H}$ be a finite-dimensional Hilbert space. At each site $\alpha \in Z^\nu$ there is a copy $\mathcal{H}_\alpha$ of $\mathcal{H}$. We denote

$$\begin{aligned} \mathcal{H}_\Lambda &= \bigotimes_{\alpha \in \Lambda} \mathcal{H}_\alpha \\ \mathcal{A}_\Lambda &= \{A: A \text{ is a self-adjoint operator on } \mathcal{H}_\Lambda\} \qquad (1.1) \\ \mathrm{tr}_\Lambda(A) &= \frac{1}{\dim \mathcal{H}_\Lambda} \mathrm{Tr}_{\mathcal{H}_\Lambda}(A) \end{aligned}$$

where $\mathrm{Tr}_{\mathcal{H}_\Lambda}(A)$ is the trace of A on $\mathcal{H}_\Lambda$ and $\dim \mathcal{H}_\Lambda$ the dimension of $\mathcal{H}_\Lambda$. In our formalism, an interaction Φ assigns to each nonempty finite subset X of Z^ν a self-adjoint operator $\Phi(X)$ on $\mathcal{H}_X$. Let $\Lambda_1 \cap \Lambda_2 = \phi$. Then $\mathcal{H}_{\Lambda_1 \cup \Lambda_2}$ can be identified naturally with $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$. We shall also identify any operator A_1 on $\mathcal{H}_{\Lambda_1}$ with the operator $A_1 \otimes \mathbb{1}$ on $\mathcal{H}_{\Lambda_1 \cup \Lambda_2}$. In particular, for any $X \subset \Lambda$, $\Phi(X)$ is identified with an operator on $\mathcal{H}_\Lambda$.

Throughout this paper we assume that the interactions we consider satisfy the following condition: There exists a constant $\alpha > 0$ such that

$$\sup_{x \in Z^\nu} \sum_{\substack{x \in X \subset Z^\nu \\ X \text{ finite}}} \|\Phi(X)\| e^{\alpha|X|} < \infty \qquad (1.2)$$

where $\|A\|$ is the norm of A and $|X|$ the cardinal number of X. Notice that finite-body interactions of short range automatically satisfy (1.2). Let

$\mathrm{dia}(X)$ be the diameter of X, defined by

$$\mathrm{dia}(X) = \sup\{|x_1 - x_2| : x_1, x_2 \in X\}$$

If $\Phi(X) = 0$ for $\mathrm{dia}(X) > \delta$ for some $\delta > 0$, Φ is called a finite-range interaction. Let $V(x)$, $x \in \Lambda$, be the unitary operator corresponding to the translation by x.[14] If $\Phi(X + x) = V(x)\Phi(X)V(x)^{-1}$, Φ is called translation invariant.

The Hamiltonian for $\Lambda \subset Z^\nu$ is a self-adjoint operator on $\mathcal{H}_\Lambda$ given by

$$H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi(X) \tag{1.3}$$

We may omit the superscript Φ in the notation without any confusions. Let $F \in \mathcal{Q}_{X_0}$, for a fixed $X_0 \subset \Lambda$. The partition function and the equilibrium state are defined by

$$\begin{aligned} Z_\Lambda^\beta &= \mathrm{tr}_\Lambda(e^{-\beta H_\Lambda}) \\ \rho_\Lambda^\beta(F) &= \left(Z_\Lambda^\beta\right)^{-1} \mathrm{tr}_\Lambda(Fe^{-\beta H_\Lambda}) \end{aligned} \tag{1.4}$$

where β is the inverse of temperature.

In order to define a classical lattice system, one only need to replace

$$\left.\begin{array}{l} \mathcal{H} \\ \mathrm{Tr}_{\mathcal{H}} \\ \mathcal{H}_\Lambda \\ \mathrm{tr}_\Lambda \\ \Phi(X) \\ \|\Phi(X)\| \end{array}\right\} \text{by} \left\{\begin{array}{l} S, \text{ a compact metric space} \\ d\mu, \text{ a probability measure on } S \\ S_\Lambda = X_{\alpha \in \Lambda} S_\alpha \\ d\mu_\Lambda = \prod_{\alpha \in \Lambda} d\mu_\alpha \\ \Phi(X) \text{ a real-valued function on } S_X \\ \|\Phi(X)\|_\infty \end{array}\right. \tag{1.5}$$

in our formalism for quantum systems. From now on we further restrict our attention to quantum lattice systems. To obtain the results for classical systems from quantum systems it suffices to use the above replacements together with the DLR equation[12] in place of Araki's Gibbs condition.

We first state the main results for general lattice systems:

Theorem 1.1. Let Φ be translational invariant or of finite range and let $F \in \mathcal{Q}_{X_0}$ for a fixed $X_0 \subset \Lambda$. Then, for sufficiently small complex β, (a) there exists a constant $M(\beta) > 0$, with $M(\beta) \to 1$ as $\beta \to 0$, such that

$$|\rho_\Lambda^\beta(F)| \leqslant \|F\| M(\beta) \qquad \text{uniformly in } \Lambda$$

(b) the limit

$$\rho^\beta(F) = \lim_{\Lambda \uparrow Z^\nu} \rho_\Lambda^\beta(F)$$

exists as Λ tends to Z^ν. Furthermore $\rho^\beta(F)$ is analytic in β in a disk centered at $\beta = 0$.

Theorem 1.2. Under the conditions in Theorem 1.1, the state determined by $\rho^\beta(F)$ is the unique KMS state for the interaction Φ.

We next discuss the results on cluster properties. The interaction Φ is said to decay exponentially if there exist a positive constant m such that

$$\sup_{x \in Z^\nu} \sum_{\substack{x \in X \subset Z^\nu \\ X \text{ finite}}} \|\Phi(X)\| e^{m \operatorname{dia}(X)} e^{\alpha|X|} < \infty \tag{1.6}$$

where α is the constant given by the condition (1.2). For finite $X_1, X_2 \subset Z^\nu$ let

$$\operatorname{dist}(X_1, X_2) \equiv \inf\{|x_1 - x_2| : x_1 \in X_1, x_2 \in X_2\} \tag{1.7}$$

For $F_1 \in \mathfrak{A}_{X_1}$ and $F_2 \in \mathfrak{A}_{X_2}$ we write

$$\rho^\beta(F_1 : F_2) = \rho^\beta(F_1 F_2) - \rho^\beta(F_1)\rho^\beta(F_2) \tag{1.8}$$

where $\rho^\beta(F)$ is the equilibrium state in the thermodynamic limit.

Theorem 1.3. Let $F_1 \in \mathfrak{A}_{X_1}$ and $F_2 \in \mathfrak{A}_{X_2}$. Under the conditions stated in Theorem 1.1, the following cluster property holds:

$$\rho^\beta(F_1; F_2) \to 0 \text{ as } \operatorname{dist}(X_1, X_2) \to \infty$$

Furthermore if Φ satisfies the condition (1.6), then for any $\epsilon > 0$ there exists a constant M such that

$$|\rho^\beta(F_1; F_2)| \leqslant M e^{-(m-\epsilon)\operatorname{dist}(X_1, X_2)}$$

Remark. Since there are well-known methods[3,5,10,14] to derive the cluster properties from the convergence of the cluster expansion, we will not produce the proof of Theorem 1.3.

Finally we give a comment on the case of two-body interactions [$\Phi(X) = 0$ if $|X| \neq 2$]. If interactions are two-body interactions, one may obtain more detailed information on the region of temperature where the cluster expansion converges. An explicit expression for the region can be obtained from Section 3 and Section 4 in a fairly straightforward manner.

We now summarize the content of this paper. In Section 2 we develop the cluster expansion. Using the fundamental theorem of calculus and an inductive argument we prove that the expansion converges absolutely and uniformly in Λ for small $|\beta|$ in Section 3. In Section 4 we construct thermodynamic limit equilibrium states by using the cluster expansion and a method of integral equations. Employing Araki's Gibbs condition[1] we show in Section 5 that the equilibrium state constructed in Section 4 is the unique KMS state.

2. THE CLUSTER EXPANSION

In this section we develop a high-temperature expansion for $\rho_\Lambda^\beta(F)$. For each $X \subset \Lambda$ we assign a real number s_X, $0 \leqslant s_X \leqslant 1$. Let

$$\begin{aligned} \{s\}_{\mathcal{P}(\Lambda)} &= \{s_X : X \in \mathcal{P}(\Lambda)\} \\ \mathcal{P}(\Lambda) &= \{X : X \subset \Lambda, X \neq \phi\} \qquad \text{for } \Lambda \neq \phi \\ \mathcal{P}(\phi) &= \{\phi\} \end{aligned} \tag{2.1}$$

For a given $\{s\}_{\mathcal{P}(\Lambda)}$ we define

$$\begin{aligned} H(\{s\}_{\mathcal{P}(\Lambda)}) &= \sum_{X \subset \Lambda} s_X \Phi(X) \\ Z^\beta(\{s\}_{\mathcal{P}(\Lambda)}) &= \operatorname{tr}_\Lambda(e^{-\beta H(\{s\}_{\mathcal{P}(\Lambda)})}) \\ \rho^\beta(F)(\{s\}_{\mathcal{P}(\Lambda)}) &= \left[Z^\beta(\{s\}_{\mathcal{P}(\Lambda)})\right]^{-1} \operatorname{tr}_\Lambda(e^{-\beta H(\{s\}_{\mathcal{P}(\Lambda)})} F) \end{aligned} \tag{2.2}$$

From the above notation it follows that

$$H(\{0\}_{\mathcal{P}(\Lambda)}) = 0, \qquad H(\{1\}_{\mathcal{P}(\Lambda)}) = H_\Lambda$$
$$Z^\beta(\{1\}_{\mathcal{P}(\Lambda)}) = Z_\Lambda^\beta \qquad \text{and} \qquad \rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)}) = \rho_\Lambda^\beta(F)$$

We want to derive an expansion for $\rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})$ that converges uniformly in $\Lambda \subset Z^\nu$.

If $\Lambda \subset Z^\nu$ is finite, we may first study

$$f(\{s\}_{\mathcal{P}(\Lambda)}) = \operatorname{tr}_\Lambda(e^{-\beta H(\{s\}_{\mathcal{P}(\Lambda)})} F) \tag{2.3}$$

for $F \in \mathcal{Q}_{X_0}$, $X_0 \subset \Lambda$. We are interested in an expansion for $f(\{1\}_{\mathcal{P}(\Lambda)})$. The idea is a partial perturbation expansion in $\{s_X : X \subset \Lambda\}$. In order to define our expansion we need the following operations: For a given $x \subset \Lambda$, we define

$$\begin{aligned} \delta^X f(\{s\}_{\mathcal{P}(\Lambda)}) &= f(\{s\}_{\mathcal{P}(\Lambda)-X}, \{1\}_X) - f(\{s\}_{\mathcal{P}(\Lambda)-X}, \{0\}_X) \\ \epsilon^X f(\{s\}_{\mathcal{P}(\Lambda)}) &= f(\{s\}_{\mathcal{P}(\Lambda)-X}, \{0\}_X) \end{aligned} \tag{2.4}$$

where $\mathcal{P}(\Lambda) - X = \mathcal{P}(\Lambda) \backslash \{X\}$. For a given $\mathcal{B} \subset \mathcal{P}(\Lambda)$, let

$$\begin{aligned} \delta^{\mathcal{B}} &= \prod_{X \in \mathcal{B}} \delta^X \\ \epsilon^{\mathcal{B}} &= \prod_{X \in \mathcal{B}} \epsilon^X \end{aligned} \tag{2.5}$$

We then have the following identity:

Lemma 2.1.

$$f(\{1\}_{\mathcal{P}(\Lambda)}) = \sum_{\phi \subseteq \mathfrak{B} \subseteq \mathcal{P}(\Lambda)} \delta^{\mathfrak{B}} f(\{0\}_{\mathcal{P}(\Lambda)})$$

Proof. From the definitions in (2.4)–(2.5) it follows that

$$\begin{aligned} f(\{1\}_{\mathcal{P}(\Lambda)}) &= \prod_{X \in \mathcal{P}(\Lambda)} (\delta^X + \epsilon^X) f(\{s\}_{\mathcal{P}(\Lambda)}) \\ &= \sum_{\phi \subset \mathfrak{B} \subset \mathcal{P}(\Lambda)} \delta^{\mathfrak{B}} \epsilon^{\mathcal{P}(\Lambda)\setminus\mathfrak{B}} f(\{s\}_{\mathcal{P}(\Lambda)}) \\ &= \sum_{\phi \subset \mathfrak{B} \subset \mathcal{P}(\Lambda)} \delta^{\mathfrak{B}} f(\{s\}_{\mathfrak{B}}, \{0\}_{\mathcal{P}(\Lambda)\setminus\mathfrak{B}}) \end{aligned}$$

Since $\delta^{\mathfrak{B}} f(\{s\}_{\mathfrak{B}}, \{0\}_{\mathcal{P}(\Lambda)\setminus\mathfrak{B}}) = \delta^{\mathfrak{B}} f(\{0\}_{\mathcal{P}(\Lambda)})$ for any $\{s\}_{\mathfrak{B}}$, the lemma follows from the above relation. ■

We next decompose $\mathfrak{B}$ into connected components: For a given $\mathfrak{B} \subset \mathcal{P}(\Lambda)$, we write

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \cdots \cup \mathfrak{B}_n$$

where $\mathfrak{B}_i \cap \mathfrak{B}_j = \phi$ if $i \neq j$, and

$$\mathfrak{B}_i = \{X_{i_1}, X_{i_2}, \ldots, X_{i_m}\} \text{ is connected}$$

That is, for any $X_{i_{l_1}}$ and $X_{i_{l_n}}$, there exist $X_{i_{l_1}}, X_{i_{l_2}}, \ldots, X_{i_{l_n}}$ such that $X_{i_{l_j}} \cap X_{i_{l_{j+1}}} \neq \phi$. From Lemma 2.1 we obtain the following:

Corollary 2.2.

$$f(\{1\}_{\mathcal{P}(\Lambda)}) = \sum_{\substack{\{\mathfrak{B}_i, \ldots, \mathfrak{B}_n\}: \\ \mathfrak{B}_i \text{ connected for } \forall i \\ \mathfrak{B}_i \neq \mathfrak{B}_j \ (i \neq j) \\ \phi \subset \mathfrak{B}_i \subset \mathcal{P}(\Lambda)}} \prod_{i=1}^{n} \delta^{\mathfrak{B}_i} f(\{0\}_{\mathcal{P}(\Lambda)})$$

For $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$, there exists $l \geqslant 0$ components which we may assume without loss of generality to be $\mathfrak{B}_1, \ldots, \mathfrak{B}_l$, that have the property

$$\mathfrak{B}_i \cup \{X_0\} \qquad \text{is connected } \forall i = 1, \ldots, l$$

Equation (2.3) shows that

$$\begin{aligned} &f\left(\{s\}_{\mathfrak{B}_1}, \ldots, \{s\}_{\mathfrak{B}_n}, \{0\}_{\mathcal{P}(\Lambda)} \setminus \bigcup_{i=1}^{n} \mathfrak{B}_i\right) \\ &\quad = f\left(\{s\}_{\mathfrak{B}_1}, \ldots, \{s\}_{\mathfrak{B}_l}, \{0\}_{\mathcal{P}(\Lambda)} \setminus \bigcup_{i=1}^{l} \mathfrak{B}_i\right) \prod_{j=l+1}^{n} Z^{\beta}(\{s\}_{\mathfrak{B}_j}, \{0\}_{\mathcal{P}(\Lambda)\setminus\mathfrak{B}_j}) \end{aligned}$$

We keep $\mathfrak{B}_1, \ldots, \mathfrak{B}_l$ fixed and first do a resummation over $\mathfrak{B}_{l+1}, \ldots, \mathfrak{B}_n$. Let

$$X\left(X_0, \{\mathfrak{B}\}_{i=1}^{l}\right) = X_0 \cup \left(\bigcup_{i=1}^{l} \bigcup_{X_{ij} \in \mathfrak{B}_i} X_{ij} \right) \tag{2.6}$$

Corollary 2.3.

$$f(\{1\}_{\mathcal{P}(\Lambda)}) = \sum_{\substack{\{\mathfrak{B}_1, \ldots, \mathfrak{B}_l\}: \\ \mathfrak{B}_i \subset \mathcal{P}(\Lambda),\ \mathfrak{B}_i \text{ connected } \forall i \\ \mathfrak{B}_i \cap \mathfrak{B}_j = \phi \\ \mathfrak{B}_i \cup \{X_0\} \text{ connected } \forall i}} \left[\prod_{i=1}^{l} \delta^{\mathfrak{B}_i} f(\{0\}_{\mathcal{P}(\Lambda)}) \right] Z^{\beta}\left(\{1\}_{\mathcal{P}(\Lambda) \setminus X(X_0, \{\mathfrak{B}_i\}_{i=1}^{l})}\right)$$

Proof. From Corollary 2.2 it follows that for a given $X \subset \Lambda$

$$Z(\{1\}_{\mathcal{P}(\Lambda \setminus X)}) = \sum_{\substack{\{\mathfrak{B}_1, \ldots, \mathfrak{B}_l\}: \\ \mathfrak{B}_i \text{ connected } \forall i \\ \mathfrak{B}_i \cap \mathfrak{B}_j = \phi\ (i \neq j) \\ \phi \subset \mathfrak{B}_i \subset \mathcal{P}(\Lambda \setminus X)}} \prod_{i=1}^{l} \delta^{\mathfrak{B}_i} Z^{\beta}(\{0\}_{\mathcal{P}(\Lambda \setminus X)})$$

The corollary follows from Corollary 2.2, the factorization property of $f(\{s\}_{\mathfrak{B}_1}, \ldots, \{s\}_{\mathfrak{B}_n})$, and the above expansion. ■

Next we change the summation notation in Corollary 2.3 into a more convenient form. It is easy to check that

$$\sum_{\substack{\{\mathfrak{B}_1, \ldots, \mathfrak{B}_n\}: \\ \mathfrak{B}_i \cap \mathfrak{B}_j = \phi\ (i \neq j) \\ \mathfrak{B}_i \text{ connected},\ \mathfrak{B}_i \subset \mathcal{P}(\Lambda) \\ \mathfrak{B}_i \cup \{X_0\} \text{ connected}}} \cdots = \sum_{\substack{\phi \subset X \subset \Lambda \\ X_0 \cap X \neq \phi\ (X \neq \phi)}} \sum_{\substack{\{X_1, \ldots, X_n\} \subset \mathcal{P}(\Lambda) \\ \cup X_i = X \\ \{X_0, X_1, \ldots, X_n\} \text{ connected}}} \cdots \tag{2.7}$$

For given $X_0, X \subset \Lambda$ we define

$$K^{\beta}(X_0, X, f) = \sum_{\substack{\{X_1, \ldots, X_n\} \subset \mathcal{P}(\Lambda) \\ \cup X_i = X \\ \{X_0, X_1, \ldots, X_n\} \text{ connected}}} \left[\prod_{i=1}^{n} \delta^{X_i} f(\{0\}_{\mathcal{P}(\Lambda)}) \right] \tag{2.8}$$

From Corollary 2.3, the definitions in (2.2), (2.3), and (2.8), and the relation (2.7) we conclude the following.

Theorem 2.4. For $F \in \mathfrak{A}_{X_0}$, $X_0 \subset \Lambda$, the following identity holds:

$$\rho^{\beta}(F)(\{1\}_{\mathcal{P}(\Lambda)}) = \sum_{\substack{\phi \subseteq X \subseteq \Lambda \\ X_0 \cap X \neq \phi\ (X \neq \phi)}} K^{\beta}(X_0, X, f) \frac{Z^{\beta}(\{1\}_{\mathcal{P}(\Lambda \setminus X \cup X_0)})}{Z^{\beta}(\{1\}_{\mathcal{P}(\Lambda)})}$$

This identity is called *the cluster expansion* for general lattice systems.

3. CONVERGENCE OF THE CLUSTER EXPANSION

Our task in this section is to prove the convergence of the cluster expansion uniformly in Λ for small real β. In the next section we will extend the result to complex β, using integral equations. We also collect some basic estimates. The basic tools are the fundamental theorem of calculus and an inductive argument similar to that used in the work of Kunz.[10] We will rewrite the cluster expansion using the fundamental theorem of calculus:

$$\delta^X f(\{s\}_{\mathcal{P}(\Lambda)-X}, \{0\}_X) = \int_0^1 ds_X \frac{\partial}{\partial s_X} f(\{s\}_{\mathcal{P}(\Lambda)})$$

$$\prod_{i=1}^n \delta^{X_i} f(\{s\}_{\mathcal{P}(\Lambda)-\{X_i\}_{i=1}^n}, \{0\}_{\{X_i\}_{i=1}^n}) = \int_0^1 \cdots \int_0^1 \pi\, ds_{X_i} \prod_{i=1}^n \frac{\partial}{\partial s_{X_i}} f(\{s\}_{\mathcal{P}(\Lambda)}) \tag{3.1}$$

Therefore we have

$$\prod_{i=1}^n \delta^{X_i} f(\{0\}_{\mathcal{P}(\Lambda)}) = \int_0^1 \cdots \int_0^1 \pi\, ds_{X_i} \prod_{i=1}^n \frac{\partial}{\partial s_{X_i}} f(\{s\}_{\{X_i\}_1^n}, \{0\}_{\mathcal{P}(\Lambda \setminus \cup X_i)}) \tag{3.2}$$

We write

$$c \equiv \sup_{x \in Z^\nu} \sum_{X \ni x} \|\Phi(X)\| \tag{3.3}$$

We will need the following estimates:

Lemma 3.1. (a) For real β

$$\frac{Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X)})}{Z^\beta(\{1\}_{\mathcal{P}(\Lambda)})} \leqslant e^{c|\beta||X|}$$

(b) For complex β

$$\left|\prod_{i=1}^n \delta^{X_i} f(\{0\}_{\mathcal{P}(\Lambda)})\right| \leqslant \|F\| e^{c|\beta||\cup X_i|} \left(\prod_{i=1}^n |\beta| \, \|\Phi(X_i)\|\right)$$

We will extend Lemma 3.1(a) to complex β in the next section.

Proof of Lemma 3.1. (a) We note that for $X \subset \Lambda$

$$H_\Lambda = H_{\Lambda\setminus X} + \hat{H}_X$$

$$\hat{H}_X \equiv \sum_{\substack{Y \subset \Lambda \\ Y \cap X \neq \phi}} \Phi(Y)$$

and

$$\|\hat{H}_X\| \leqslant |X| \sup_{x\in X} \sum_{Y\ni x} \|\Phi(Y)\|$$
$$\leqslant c|X|$$

where the constant c is given in (3.3). Hence, by the Peierls–Bogoliubov inequality[14,15] one obtains

$$\begin{aligned}\operatorname{tr}(e^{-\beta H_\Lambda}) &= \operatorname{tr}(e^{-\beta H_{\Lambda\setminus X}-\beta\hat{H}_X})\\ &\geqslant \operatorname{tr}(e^{-\beta H_{\Lambda\setminus X}})e^{-|\beta|\,\|\hat{H}_X\|}\\ &\geqslant \operatorname{tr}(e^{-\beta H_{\Lambda\setminus X}})e^{-c|\beta|\,|X|}\end{aligned}$$

Since for any $X \subset \Lambda$

$$Z^\beta(\{1\}_{\mathcal{P}(X)}) = \operatorname{tr}(e^{-\beta H_X})$$

Part (a) of the lemma follows from the above inequality.

(b) We recall a useful formula: Let $B = B(\lambda)$ be a bounded operator. Then

$$\frac{d}{d\lambda} e^{B(\lambda)} = \int_0^1 ds\, e^{sB(\lambda)}B'(\lambda)e^{(1-s)B(\lambda)}$$

Next, let $B(\lambda_1, \ldots, \lambda_n)$ be a bounded operator which is linear in each λ_i. Then

$$\begin{aligned}\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n} e^{B(\lambda_1,\ldots,\lambda_n)} = \sum_{\Pi\in P_n}\int_{\Delta_n}\prod ds_i\, e^{s_1B(\lambda_1,\ldots,\lambda_n)}\\ \times\frac{\partial}{\partial\lambda_{\Pi(1)}}B(\lambda_1,\ldots,\lambda_n)e^{s_2B(\lambda_1,\ldots,\lambda_n)}\ldots\\ \times\frac{\partial}{\partial\lambda_{\Pi(n)}}B(\lambda_1,\ldots,\lambda_n)\\ \times\exp\left[\left(1-\sum_{i=1}^n s_i\right)B(\lambda_1,\ldots,\lambda_n)\right]\end{aligned} \tag{3.4}$$

where $\Delta_n = \{s_1, \ldots, s_n : 0 \leqslant s_i,\ \sum_{i=1}^n s_i = 1\}$ and P_n is the permutation group of $\{1, \ldots, n\}$. Hence

$$\left\|\frac{\partial^n}{\partial\lambda_1\ldots\partial\lambda_n} e^{B(\cdots)}\right\| \leqslant n!\,|\Delta_n|\left(\prod_{i=1}^n\left\|\frac{\partial}{\partial\lambda_i}B(\cdots)\right\|\right)e^{\|B(\cdots)\|} \tag{3.5}$$

and $|\Delta_n| = 1/n!$. We apply the above result. It follows from (3.2) and (3.5) that

$$\left|\prod_{i=1}^{n}\delta^{X_i}f(\{0\}_{\mathcal{P}(\Lambda)})\right| \leqslant \int_0^1\cdots\int_0^1 \pi\, ds_{X_i}\left|\prod_{i=1}^{n}\frac{\partial}{\partial s_{X_i}}f(\{s\}_{\{X_i\}_1^n},\{0\})\right|$$

$$\leqslant \left|\prod_{i=1}^{n}\frac{\partial}{\partial s_{X_i}}\operatorname{tr}\left(e^{-\beta H(\{s\}_{\{X_i\}_1^n},\{0\})}F\right)\right|$$

$$\leqslant \left(\prod_{i=1}^{n}|\beta|\left\|\frac{\partial}{\partial s_{X_i}}H(\{s\}_{\{X_i\}_1^n},\{0\})\right\|\right)e^{|\beta|\,\|H(\{s\}_{\{X_i\}_1^n})\|}\|F\|$$

$$\leqslant \left(\prod_{i=1}^{n}|\beta|\,\|\Phi(X_i)\|\right)e^{c|\beta|\,|\cup X_i|}\|F\|$$

Here we have used the fact that for $X = \cup X_i$

$$\|H(\{s\}_{\{X_i\}_1^n},\{0\})\| \leqslant |X|\sup_{x\in X}\sum_{Y\ni x}\|\Phi(Y)\|$$
$$\leqslant c|X|$$

This completes the proof of Lemma 3.2. ■

For finite X_0, $X \subset Z^\nu$ we define

$$F^\beta(X_0,X;\Phi) = \sum_{\substack{\{X_1,\ldots,X_n\}\subset\mathcal{P}(X)\\ \cup X_i = X\\ \{X_0,X_1,\ldots,X_n\}\text{ connected}}}\left(\prod_{i=1}^{n}|\beta|\,\|\Phi(X_i)\|\right) \tag{3.6}$$

We now combine Theorem 2.4, Eq. (3.2), and Lemma 3.1 to conclude that

$$\left|\rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})\right| \leqslant \|F\|\sum_{\substack{\phi\subset X\subset\Lambda\\ X\cap X_0\neq\phi\ (X\neq\phi)}} e^{2c|\beta|(|X_0\cup X|)}F^\beta(X_0,X;\Phi) \tag{3.7}$$

The term corresponding to $X=\phi$ in the summation $\sum_X$ is 1. In order to show the convergence of the cluster expansion we have to bound the right-hand side of (3.7) uniformly in Λ for sufficiently small β.

We write

$$f_\beta(X) = |\beta|\,\|\Phi(X)\|$$
$$f_\beta(\phi) = 1 \tag{3.8}$$

Let $X\neq\phi$ and Y be finite subsets in Z^ν. For $X\cap Y=\phi$, we define

$$A(X,Y) = \sum_{\substack{\{Y_1,\ldots,Y_n\}\subset\mathcal{P}(X\cup Y):\\ Y\subset\cup Y_i\subset X\cup Y\\ \{X,Y_1,\ldots,Y_n\}\text{ connected}}}\prod_{i=1}^{n}f_\beta(Y_i) \tag{3.9}$$

$$I(m,n) = \sup_{\substack{X\\ |X|=m}}\sum_{\substack{Y\\ |Y|=n}}A(X,Y) \tag{3.10}$$

The following is one of the basic estimates in this paper:

Proposition 3.2. For sufficiently small $|\beta|$, there exist constant c_1 and $A(\beta)$, with $A(\beta)\to 1$ as $|\beta|\to 0$, such that

$$I(m,n) \leqslant |\beta| c_1 A(\beta)^{m+n} e^{c|\beta| m} e^{-\alpha n/4}$$

where the constant α is determined by the condition (1.2).

Proof. We prove the proposition by an induction with respect to m and n. Let $X' = X - \{x\}$ for a fixed $x \in X$. We isolate the contribution of $f_\beta(X_i)$ for the set X_i containing x in $A(X,Y)$. From the definition of $A(X,Y)$ in (3.9) it follows that

$$A(X,Y) = \sum_{\substack{\phi \subset W \subset X \cup Y \\ x \in W\ (W \neq \phi)}} \left[\sum_{\substack{\{W_1,\dots,W_n\} \subset \mathcal{P}(W) \\ \cup W_i = W \\ x \in W_i\ (W_i \neq \phi)}} \prod_{i=1}^{n} f_\beta(W_i) \right] A(X' \cup W', Y \setminus W') \tag{3.11}$$

where the contribution corresponding to the case of $W = \phi$ is $A(X',Y)$. We break the sum over W into two parts:

$$\sum_W = \sum_{W \subset X} + \sum_{W \cap Y \neq \phi}$$

Let $x + X = \{x\} \cup X$ for any $X \subset Z^\nu$. Then it follows from (3.11) that

$$\begin{aligned} A(X,Y) = &\left\{ \sum_{\substack{\phi \subset \bar{X} \subset X: \\ x \in \bar{X}\ (\bar{X} \neq \phi)}} \sum_{\substack{\{\bar{X}_1,\dots,\bar{X}_m\} \subset \mathcal{P}(\bar{X}): \\ \cup \bar{X}_i = \bar{X} \\ x \in \bar{X}_i\ (\bar{X}_i \neq \phi)}} \prod_{i=1}^{m} f_\beta(\bar{X}_i) \right\} \\ &\times \left\{ A(X',Y) + \sum_{\phi \neq T \subseteq Y} \left[\sum_{\substack{\{T_1,\dots,T_l\} \subset \mathcal{P}(T \cup X'): \\ T \subset \cup T_i \subset T \cup X' \\ T_i \cap T \neq \phi}} \prod_{i=1}^{l} f_\beta(x + T_i) \right] \right. \\ &\qquad \left. \times A(X' \cup T, Y \setminus T) \right\} \end{aligned} \tag{3.12}$$

In the above expression, $A(X',Y) = 0$ if $X' = \phi$ and the summation over T is zero if $Y = \phi$.

We first note that, since $f_\beta \geqslant 0$,

$$\sum_{\substack{\phi \subset \bar{X} \subset X \\ x \in \bar{X}\ (\bar{X} \neq \phi)}} \sum_{\substack{\{\bar{X}_1, \ldots, \bar{X}_m\} \subset \mathcal{P}(\bar{X}) \\ \cup \bar{X}_i = \bar{X} \\ x \in \bar{X}_i\ (\bar{X}_i \neq \phi)}} \prod_{i=1}^{m} f_\beta(\bar{X}_i) \leqslant \sum_{m=0}^{\infty} \frac{1}{m!} \Bigg[\sum_{\substack{Z \subset X: \\ x \in Z}} f_\beta(Z) \Bigg]^m$$

$$\leqslant \sum_{m=0}^{\infty} \frac{1}{m!} \Bigg(|\beta| \sum_{\substack{Z \subset Z^\nu \\ x \in Z}} \|\Phi(Z)\| \Bigg)^m$$

$$\leqslant e^{c|\beta|} \tag{3.13}$$

We define

$$b(k) \equiv \sum_{T : |T| = k} \sum_{\substack{\{T_1, \ldots, T_l\} \subset \mathcal{P}(T \cup X') \\ T \subset \cup T_i \subset T \cup X' \\ T_i \cap T \neq \phi}} \prod_{i=1}^{l} f_\beta(x + T_i) \tag{3.14}$$

From condition (1.2) we obtain the following bound:

$$b(k) \leqslant e^{-\alpha k/2} \sum_{T : |T| = k} \sum_{\substack{\{T_1, \ldots, T_l\} \subset \mathcal{P}(T \cup X') \\ T \subset \cup T_i \subset T \cup X' \\ T_i \cap T = \phi}} \prod_{i=1}^{l} e^{\alpha|x + T_i|/2} f_\beta(x + T_i)$$

$$\leqslant e^{-\alpha k/2} \sum_{l=1}^{\infty} \frac{1}{l!} \Bigg[\sum_{\phi \subset Z \subset Z^\nu} |\beta| \, \|\Phi(x + Z) e^{\alpha|x+Z|/2} \Bigg]^l$$

$$\leqslant e^{-\alpha k/2} (e^{|\beta| c'} - 1)$$

$$\leqslant e^{-\alpha k/2} |\beta| c_1 \tag{3.15}$$

for some constant $c_1(\alpha)$. Combining (3.9)–(3.15) we conclude that

$$I(m,n) \leqslant e^{c|\beta|} \Bigg[I(m-1, n) + |\beta| c_1 \sum_{k=1}^{n} e^{-\alpha k/2} I(m - 1 + k, n - k) \Bigg] \tag{3.16}$$

Notice that $I(0.1) = 0$ by the definition of $A(X, Y)$, and

$$I(1,0) = \sup_{x \in Z^\nu} |\beta| \, \|\Phi(x)\|$$

$$\leqslant e^{c|\beta|}$$

$$I(1,1) = \sup_{\substack{x, y \in Z^\nu \\ x \neq y}} \big[1 + f_\beta(x) \big] f_\beta(x, y)$$

$$\leqslant (1 + |\beta| c) |\beta| c$$

$$\leqslant e^{c|\beta|} c_1 |\beta| e^{-\alpha/4} \tag{3.17}$$

for some constant $c_1(\alpha)$. By choosing

$$A(\beta) = 1 + c_1|\beta| \sum_{k=1}^{\infty} e^{-(\alpha/4 - c|\beta|)k} \tag{3.18}$$

which converges for $|\beta| \leqslant \alpha/4c$ and tends to 1 as $|\beta| \to 0$, the proposition follows by induction from (3.16) and (3.17). ∎

We now prove the convergence of the cluster expansion:

Theorem 3.3. For sufficiently small real β, the cluster expansion converges absolutely and uniformly in Λ. Furthermore there exists a constant $B(\beta)$ such that for $F \in \mathcal{Q}_{X_0}$

$$\left|\rho^{\beta}(F)(\{1\}_{\mathcal{P}(\Lambda)})\right| \leqslant \|F\| e^{4c|X_0|} B(\beta)$$

where $B(\beta) \to 1$ as $|\beta| \to 0$.

Proof. In (3.7) we break the sum over X into two terms:

$$\sum_{\substack{\phi \subseteq X \subseteq \Lambda \\ X \cap X_0 \neq \phi \ (X \neq \phi)}} = \sum_{\phi \subseteq X \subseteq X_0} + \sum_{\substack{X \not\subset X_0 \\ X \cap X_0 \neq \phi}} \tag{3.19}$$

The first sum is bounded by

$$\begin{aligned}
&\sum_{\phi \subseteq X \subseteq X_0} e^{2c|\beta|(|X_0 \cup X|)} \sum_{\substack{\{X_1, \ldots, X_n\} \subset \mathcal{P}(X) \\ \cup X_i = X}} \prod_{i=1}^{n} f_\beta(X_i) \\
&\leqslant e^{3c|\beta|\,|X_0|} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{Z \subset X_0} f_\beta(Z) \right]^n \\
&\leqslant e^{3c|\beta|\,|X_0|} \sum_{n=0}^{\infty} \frac{1}{n!} \left[|X_0| \sup_{x \in X_0} \sum_{Z \ni x} f_\beta(Z) \right]^n \\
&\leqslant e^{4c|\beta|\,|X_0|}
\end{aligned} \tag{3.20}$$

by (3.14). Let $Y = X \backslash X_0$. Then the second term in (3.19) is bounded by

$$\begin{aligned}
&e^{2c|\beta|\,|X_0|} \sum_{\substack{X \not\subset X_0 \\ X \cap X_0 \neq \phi}} e^{2c|\beta|\,|X \backslash X_0|} \sum_{\substack{\{X_1, \ldots, X_n\} \subset \mathcal{P}(X) \\ \cup X_i = X \\ \{X_0, X_1, \ldots, X_n\} \text{ connected}}} \prod_{i=1}^{n} f_\beta(X_i) \\
&\leqslant e^{2c|\beta|\,|X_0|} \sum_{\phi \neq Y} e^{2c|\beta|\,|Y|} \sum_{\substack{\{Y_1, \ldots, Y_n\}: \\ Y \subset Y_i \subset X_0 \cup Y \\ Y_i \neq Y_j \ (i \neq j)}} \prod_{i=1}^{n} f_\beta(Y_i) \\
&\leqslant e^{2c|\beta|\,|X_0|} \sum_{\phi \neq Y} e^{2c|\beta|\,|Y|} A(X_0, Y)
\end{aligned} \tag{3.21}$$

by the definition of $A(X, Y)$ in (3.9). From (3.10) and Proposition 3.2 it follows that

$$\sum_{\phi \neq Y} e^{2c|\beta||Y|} A(X_0, Y) \leqslant \sum_{n=1}^{\infty} e^{2c|\beta||Y|} I(|X_0|, n)$$
$$\leqslant |\beta| c_1 A(\beta)^{|X_0|} e^{c|\beta||X_0|} \sum_{n=1}^{\infty} \left[A(\beta) e^{-(\alpha/4 - 2|\beta|)}\right]^n$$
$$\leqslant |\beta| c_1 c_2 A^{|X_0|} e^{c|\beta||X_0|} \tag{3.22}$$

for small β such that $A(\beta)e^{-(\alpha/4-2|\beta|)} < 1$. By choosing

$$B(\beta) = 1 + |\beta| c_1 c_2 A(\beta)^{|X_0|}$$

the theorem follows from (3.7) and (3.20)–(3.22). ■

4. INTEGRAL EQUATIONS AND THERMODYNAMIC LIMITS OF EQUILIBRIUM STATES

In this section we prove the existence of the thermodynamic limit for $\rho_\Lambda^\beta(F)$ for sufficiently small complex β, and establish the analyticity of infinite volume expectations. We will combine the cluster expansion with a method of integral equations of Kirkwood–Salsburg type.[14] We first state the main result in this section:

Theorem 4.1. Let Φ be translation invariant or of finite range and let $F \in \mathcal{Q}_{X_0}$ for a finite $X_0 \subset Z^\nu$. Then for sufficiently small complex β, (a) there exists a constant $M(\beta)$ such that $M(\beta) \to 1$ as $\beta \to 0$ and

$$|\rho_\Lambda^\beta(F)| \leqslant \|F\| M(\beta), \text{ uniformly in } \Lambda$$

(b) the limit

$$\rho^\beta(F) = \lim_{\Lambda \uparrow Z^\nu} \rho_\Lambda^\beta(F)$$

exists and is analytic in β, for $|\beta|$ sufficiently small.

At the end of this section we will give the region of convergence for two body interactions explicitly.

For $X \subset \Lambda$ we define

$$g_\Lambda^\beta(X) \equiv \frac{Z^\beta(\{1\}_{\mathcal{P}(\Lambda \setminus X)})}{Z^\beta(\{1\}_{\mathcal{P}(\Lambda)})} \tag{4.1}$$

A priori we know that $g_\Lambda^\beta(X)$ is defined for real β. In the later part of this section we will show that $g_\Lambda^\beta(X)$ can be extended to complex β via a method of integral equations if $|\beta|$ is sufficiently small. In order to prove

Theorem 4.1 we need the following result:

Proposition 4.2. Under the conditions in Theorem 4.1, (a) there exists a constant c independent of β and Λ such that

$$|g_\Lambda^\beta(X)| \leqslant e^{c|\beta||X|}$$

Furthermore $g_\Lambda^\beta(X)$ is analytic in the region of small β. (b) The limit

$$g^\beta(X) = \lim_{\Lambda \uparrow Z^\nu} g_\Lambda^\beta(X)$$

exists and is analytic in the region of small $|\beta|$.

We first show Theorem 4.1 by applying Proposition 4.2.

Proof of Theorem 4.1. (a) This follows from Theorem 3.3 and Proposition 4.2 (a). (b) Since $g_\Lambda^\beta(X)$ is analytic in the region of small β by Proposition 4.2, $\rho^\beta(\{1\}_{\mathcal{P}(\Lambda)})$ defined by the cluster expansion in Theorem 2.4 is also analytic. Therefore, by Vitali's theorem and Theorem 4.1(a) it is sufficient to show that for small real β the limit

$$\rho^\beta(F) = \lim_{\Lambda \uparrow Z^\nu} \rho_\Lambda^\beta(F) \tag{4.2}$$

exists as Λ tends to Z^ν. By Theorem 3.3 the cluster expansion for $\rho_\Lambda^\beta(F) = \rho^\beta(\{1\}_{\mathcal{P}(\Lambda)})$ is absolutely summable, uniformly in Λ for small real β. Hence to prove (4.2) it suffices to show that each term in the expansion converges as $\Lambda \to Z^\nu$. For given X (finite), the expression $K^\beta(X_0, X; f)$ defined in (2.8) is independent of Λ for sufficiently large Λ. Since $g_\Lambda^\beta(X)$ converges as $\Lambda \to Z^\nu$ by Proposition 4.2(b), each term in the cluster expansion converges as $\Lambda \to Z^\nu$. This completes the proof of Theorem 4.1. ∎

In the rest of this section we prove Proposition 4.2 by using integral equations. The equations we are considering are of the Kirkwood–Salsburg type. We first consider $g_\Lambda^\beta(X)$ defined by (4.1) for real β, which is well defined. We want to derive an integral equation for $g_\Lambda^\beta(X)$. From now on we suppress the superscript β in $g_\Lambda^\beta(X)$.

Let f be a function defined on the set of finite subsets of Z^ν. Such functions form a Banach space $\mathcal{F}_\xi$:

$$\mathcal{F}_\xi = \left\{ f : \|f\| = \sup_X \xi^{-|X|} |f(X)| < \infty, \xi > 0 \right\} \tag{4.3}$$

We propose to derive an equation of the form

$$\begin{aligned} g_\Lambda &= \mathbb{1} + K_\Lambda g_\Lambda \\ g &= \mathbb{1} + K g \end{aligned} \tag{4.4}$$

where $\mathbb{1}(\phi) = 1$ and $\mathbb{1}(X) = 0$, with $\|K_\Lambda\| < 1$ uniformly in Λ and

$$\|K_\Lambda - K\| \to 0 \quad \text{as} \quad \Lambda \to Z^\nu$$

Then it follows that

$$g_\Lambda = (1 - K_\Lambda)^{-1}\mathbb{1}$$

$$g_\Lambda \to g \quad \text{as} \quad \Lambda \to Z^\nu$$

For details we refer the reader to Refs. 13 and 14.

We now apply the cluster expansion to derive (4.4). We fix a point $x_0 \in X$. Then, since

$$Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X - x_0)}) = \left[\prod_{\substack{Y \subset \Lambda\setminus X - x_0: \\ x_0 \in Y}} (\delta^Y + \epsilon^Y)\right] Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X - x_0)})$$

$$= Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) + f_\Lambda(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) \tag{4.5}$$

where

$$f_\Lambda(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) = \left\{\sum_{\substack{\{Y_1, \ldots, Y_n\}: \\ \phi \neq Y_i \neq Y_j\ (i \neq j) \\ x_0 \in Y_i \subset \Lambda\setminus X - x_0}} \prod_{i=1}^n \delta^{Y_i}\right\} Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X - x_0)}) \tag{4.6}$$

it follows that

$$Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) = Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X - x_0)}) - f_\Lambda(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) \tag{4.7}$$

We apply the cluster expansion to f. The role of Λ is now played by $\Lambda\setminus X$;

$$f_\Lambda(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) = \left\{\sum_{\substack{\{Y_1, \ldots, Y_n\}: \\ \phi \neq Y_i \neq Y_j\ (i \neq j) \\ x_0 \in Y_i \subset \Lambda\setminus X - x_0}} \prod_{i=1}^n \delta^{Y_i}\right\} \prod_{\overline{W} \subset \Lambda\setminus X} (\delta^W + \epsilon^W) Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X - x_0)})$$

Following the process used to prove Corollary 2.3 and Theorem 2.4 we arrive at the following identity:

$$f_\Lambda(\{1\}_{\mathcal{P}(\Lambda\setminus X)}) = \sum_{\substack{\phi \neq S \subset \Lambda\setminus X - x_0 \\ x_0 \in S}} K^\beta(\{x_0\}, S; Z^\beta) Z^\beta(\{1\}_{\mathcal{P}(\Lambda\setminus X \cup S)}) \tag{4.8}$$

where $K^\beta(\{x_0\}, S; Z^\beta)$ has been defined in (2.8). For any complex β and $f \in \mathcal{F}_\xi$ we define an operator K on $\mathcal{F}_\xi$ by

$$\begin{aligned} &(Kf)(\phi) = 0 \\ &(Kf)(X) = f(X - x_0) - \sum_{\substack{\phi \neq S \subset Z^\nu\setminus X - x_0 \\ x_0 \in S}} K^\beta(\{x_0\}, S; Z^\beta) f(X \cup S) \end{aligned} \tag{4.9}$$

for $X \neq \phi$. We introduce the operator χ_Λ on $\mathcal{F}_\xi$ defined by

$$(\chi_\Lambda f)(X) = \chi_\Lambda(X) f(X) \qquad \text{for all} \quad f \in \mathcal{F}_\xi \tag{4.10}$$

where $\chi_\Lambda(X) = 1$ if $X \subset \Lambda$ and $\chi_\Lambda(X) = 0$ otherwise. Let $\mathbb{1}$ be the element in $\mathcal{F}_\xi$ defined by $\mathbb{1}(\phi) = 1$ and $\mathbb{1}(X) = 0$ otherwise. Then, from the definition of K in (4.9) and the fact that $g_\Lambda(\phi) = 1$ we obtain the following lemma.

Lemma 4.3.

$$g_\Lambda = \mathbb{1} + \chi_\Lambda K \chi_\Lambda g_\Lambda \tag{4.11}$$

The above relation is the integral equation of the Kirkwood–Salsburg type we want to derive. Beside (4.11) let us consider also the equation

$$g = \mathbb{1} + Kg \tag{4.12}$$

In both (4.11) and (4.12) we will allow β to be complex, using the following proposition.

Proposition 4.4. For sufficiently small complex β and for $\xi = e^{\alpha/8}$,

$$\|\chi_\Lambda K \chi_\Lambda\| < 1 \qquad \text{uniformly in } \Lambda$$

and

$$\|K\| < 1$$

Proof. From the definition in (4.9) and (4.10) we get

$$\begin{aligned}\|\chi_\Lambda K \chi_\Lambda\| &\leqslant e^{-\alpha/8} + \sup_{x_0 \in Z^\nu} \sum_{x_0 \in S} \left| K^\beta\big(\{x_0\}, S; Z^\beta\big)\right| e^{\alpha|S|/8} \\ &\leqslant e^{-\alpha/8} + \sup_{x_0 \in Z^\nu} \sum_{x_0 \in S^\nu} e^{(c|\beta| + \alpha/8)|S|} F^\beta\big(\{x_0\}, S; \Phi\big)\end{aligned} \tag{4.13}$$

where the quantity $F^\beta(\{x_0\}, S; \Phi)$ has been defined in (3.6). Here we have used Lemma 3.1(b) to get the second inequality. Using the notations in (3.8)–(3.10) we obtain

$$\|\chi_\Lambda K \chi_\Lambda\| \leqslant e^{-\alpha/8} + |\beta| \, \|\Phi(x_0)\| e^{(c|\beta| + \alpha/8)} + \sum_{n=1}^{\infty} e^{[c|\beta| + \alpha/8]n} I(1, n) \tag{4.14}$$

We apply Proposition 3.2 to conclude that

$$\|\chi_\Lambda K \chi_\Lambda\| < 1$$

for small complex β. To prove $\|K\| < 1$ we only note that $\|K\|$ is bounded by the right-hand side of (4.13). This completes the proof of the proposition. ■

We finally prove Proposition 4.2.

Proof of Proposition 4.2. (a) Since g_Λ satisfy (4.11) for real β,

$$g_\Lambda = (1 - \chi_\Lambda K \chi_\Lambda)^{-1} \mathbb{1} \tag{4.15}$$

for small real β. Furthermore, since $\chi_\Lambda K \chi_\Lambda$ is analytic in β, one may extend (4.15) to complex β, for small $|\beta|$. Notice that

$$\| g_\Lambda \| = \sup_X e^{-\alpha |X|/8} | g_\Lambda |$$

and

$$\begin{aligned} \| g_\Lambda \| &\leqslant \left\| (1 - \chi_\Lambda K \chi_\Lambda)^{-1} \mathbb{1} \right\| \\ &\leqslant \text{const} \end{aligned}$$

for small $|\beta|$. This proves part (a) of the proposition.

(b) Using an argument in Ruelle,[14] part (b) of the proposition will be proved if we can show that for $\Lambda \subset \Lambda' \subset \Lambda''$

$$\| \chi_\Lambda K \chi_{\Lambda''} - \chi_\Lambda K \chi_{\Lambda'} \| \leqslant \eta(\delta)$$

where δ is the minimal distance from Λ to the boundary of Λ' and $\eta(\delta)$ satisfies $\lim_{\delta \to \infty} \eta(\delta) = 0$. This inequality follows from

$$\begin{aligned} &\left| \chi_\Lambda(X)(K\chi_{\Lambda''} f)(X) - \chi_\Lambda(X)(K\chi_{\Lambda'} f)(X) \right| \\ &\qquad \leqslant \sum_{\substack{S \subset Z^\nu \setminus X - x_0 \\ x_0 \in S \not\subset \Lambda' \\ X \subset \Lambda}} \left| K^\beta(\{x_0\}, S; Z^\beta) \right| \left| f(X \cup S) \right| \end{aligned}$$

From this and Lemma 3.1(b) we get

$$\begin{aligned} &\| \chi_\Lambda K \chi_{\Lambda''} - \chi_\Lambda K \chi_{\Lambda'} \| \leqslant \eta(\delta) \\ &\qquad \eta(\delta) \leqslant \sup_{x_0 \in \Lambda} \sum_{x_0 \in S \not\subset \Lambda'} F^\beta(\{x_0\}, S; \Phi) e^{(c|\beta| + \alpha/8)|S|} \end{aligned} \tag{4.16}$$

If Φ has finite range, i.e., $\Phi(X) = 0$ if $\text{dia}(X) \geqslant \delta_0$ for fixed δ_0, then the number of Φ's in $F^\beta(\{x_0\}, S; \Phi)$ is greater than δ/δ_0 [see the definition of F^β in (3.6)]. Thus it follows that $|S| \geqslant \delta/\delta_0$ and

$$\eta(\delta) \leqslant \sum_{l \geqslant \delta/\delta_0}^{\infty} e^{[c|\beta| + \alpha/8]l} I(1, l)$$

by the argument used in (4.14). From Proposition 3.2 we conclude that $\eta(\delta) \to 0$ as $\delta \to \infty$ for small β.

On the other hand if Φ is translation invariant,

$$\eta(\delta) = \sum_{\substack{x_0 \in S \not\subset B_\delta(x_0) \\ x_0 \text{ fixed}}} e^{[c|\beta| + \alpha/8]|S|} F^\beta(\{x_0\}, S; \Phi)$$

where $B_\delta(x_0)$ is the ball of radius δ centered at x_0.

Since

$$\sum_{\substack{x_0 \in S \\ x_0 \text{ fixed}}} \cdots$$

is summable, see (4.13)–(4.14), we conclude that $\eta(\delta) \to 0$ as $\delta \to \infty$. This completes the proof. ■

5. UNIQUENESS OF KMS STATES

Our next task is to show that the state defined by the limit

$$\rho^\beta(F) = \lim_{\Lambda \uparrow Z^\nu} \rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})$$

is the unique KMS state for a given interaction Φ. In order to show the uniqueness we must show that any KMS state corresponding to Φ coincides with $\rho^\beta(F)$ for small β. Let $\bar{\rho}^\beta(F)$ be a KMS state for given Φ and α_t the time translation automorphism corresponding to Φ.(14,16) Let Ω^β be the cyclic vector corresponding to $\bar{\rho}^\beta(F)$ and h the generator of the modular automorphism group corresponding to $(\bar{\rho}^\beta, \alpha_t)$.(14,16) We define

$$\begin{aligned}
H_\Lambda(\Phi) &= \sum_{X \subset \Lambda} \Phi(X) \\
W_{\Lambda,\Lambda^c} &= \sum_{\substack{X: \\ X \cap \Lambda \neq \phi \\ X \cap \Lambda^c \neq \phi}} \Phi(X) \\
k(\Lambda) &= H_\Lambda + W_{\Lambda,\Lambda^c} \\
h(\Lambda) &= h - k(\Lambda) \\
U_{k(\Lambda)}(s) &= e^{s\beta[h - k(\Lambda)]} = e^{s\beta h(\Lambda)}, \qquad 0 \leqslant s \leqslant 1
\end{aligned} \tag{5.1}$$

We denote for finite Λ, $X \subset Z^\nu$

$$\begin{aligned}
\bar{\mathcal{P}}(\Lambda, X) &= \{ Y : Y \cap \Lambda \neq \phi, Y \cap X = \phi \} \\
\bar{\mathcal{P}}(\Lambda) &= \bar{\mathcal{P}}(\Lambda, \phi)
\end{aligned} \tag{5.2}$$

Let $\bar{\rho}^\beta(F)(\{0\}_{\bar{\mathcal{P}}(\Lambda)})$ be the state on $\mathcal{C}_{X_0}$ defined by

$$\bar{\rho}^\beta(F)(\{0\}_{\bar{\mathcal{P}}(\Lambda)}) = N^{-1}\left(U_{k(\Lambda)}(\tfrac{1}{2})\Omega^\beta, F U_{k(\Lambda)}(\tfrac{1}{2})\Omega^\beta\right) \tag{5.3}$$

where N is the normalization factor. We now recall Araki's Gibbs condition [1]:

$$\bar{\rho}^\beta(\cdot)(\{0\}_{\bar{\mathcal{P}}(\Lambda)}) = \operatorname{tr}_\Lambda(\cdot) \otimes \varphi_{\Lambda^c}(\cdot) \tag{5.4}$$

where φ_{Λ^c} is a KMS state on $\mathfrak{A}_{\Lambda^c}$. We now construct $\bar{\rho}^\beta(F) = \bar{\rho}^\beta(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)})$ from $\bar{\rho}^\beta(F)(\{0\}_{\bar{\mathcal{P}}(\Lambda)})$ by means of the cluster expansion. We denote the cyclic vector corresponding to $\bar{\rho}^\beta(\cdot)(\{0\}_{\bar{\mathcal{P}}(\Lambda)})$ by $\Omega^\beta(\{0\}_{\bar{\mathcal{P}}(\Lambda)})$ and $\Omega^\beta = \Omega^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda)})$. We define

$$\begin{aligned} k(\{s\}_{\bar{\mathcal{P}}(\Lambda)}) &= \sum_{X\,:\,X\cap\Lambda\neq\phi} S_X\Phi(X) \\ h(\{s\}_{\bar{\mathcal{P}}(\Lambda)}) &= h(\Lambda) + k(\{s\}_{\bar{\mathcal{P}}(\Lambda)}) \end{aligned} \tag{5.5}$$

We set

$$\begin{aligned} \bar{Z}^\beta(\{s\}_{\bar{\mathcal{P}}(\Lambda)}) &= \left(\Omega^\beta(\{0\}_{\bar{\mathcal{P}}(\Lambda)}), e^{-\beta h(\{s\}_{\bar{\mathcal{P}}(\Lambda)})}\Omega^\beta(\{0\}_{\bar{\mathcal{P}}(\Lambda)})\right) \\ \bar{f}(\{s\}_{\bar{\mathcal{P}}(\Lambda)}) &= \left(e^{-\beta h(\{s\}_{\bar{\mathcal{P}}(\Lambda)})/2}\Omega^\beta(\{0\}_{\bar{\mathcal{P}}(\Lambda)}),\right. \\ &\qquad \left.\times F e^{-\beta h(\{s\}_{\bar{\mathcal{P}}(\Lambda)})/2}\Omega^\beta(\{0\}_{\bar{\mathcal{P}}(\Lambda)})\right) \end{aligned} \tag{5.6}$$

Clearly we have that for $F \in \mathfrak{A}_{X_0}$, $X_0 \subset \Lambda$,

$$\bar{\rho}^\beta(F) = \frac{\bar{f}(\{1\}_{\bar{\mathcal{P}}(\Lambda)})}{Z^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda)})} \qquad \text{for any } \Lambda \tag{5.7}$$

We want to show that for small β

$$\bar{\rho}^\beta(F) = \rho^\beta(F) \tag{5.8}$$

where $\rho^\beta(F)$ is the state given by the limit

$$\rho^\beta(F) = \lim_{\Lambda\uparrow Z^\nu} \rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})$$

in the previous section. That is, $\bar{\rho}^\beta(F)$ is unique and is independent of boundary conditions.

To prove (5.8) we apply the cluster expansion to $\bar{\rho}^\beta(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)})$ for $F \in \mathfrak{A}_{X_0}$, $X_0 \subset \Lambda$:

$$\bar{f}(\{1\}_{\bar{\mathcal{P}}(\Lambda)}) = \sum_{\substack{\{\mathfrak{B}_1,\ldots,\mathfrak{B}_l\}: \\ \mathfrak{B}_i \text{ connected}, \mathfrak{B}_i \subset \bar{\mathcal{P}}(\Lambda) \\ \mathfrak{B}_i \cap \mathfrak{B}_j = \phi (i\neq j) \\ \mathfrak{B}_i \cup \{X_0\} \text{ connected}}} \left[\prod_{i=1}^{l} \delta^{\mathfrak{B}_i}\bar{f}(\{0\}_{\bar{\mathcal{P}}(\Lambda)})\right] Z^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda, X(X_0,\{\mathfrak{B}_i\}))})$$

Here we use the factorization property of $\bar{Z}^\beta$ in (5.3) and the method used in the derivation of Corollary 2.3. Dividing by $\bar{Z}^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda)})$ and applying (2.7) again we have the expansion

$$\bar{\rho}^\beta(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)}) = \sum_{\substack{\phi\subseteq X: \\ X\cap\Lambda\neq\phi \\ X\cap X_0\neq\phi}} K^\beta(X_0, X; \bar{f})\bar{g}_\Lambda(X \cup X_0) \tag{5.9}$$

where

$$\bar{g}_\Lambda(X) = \frac{\bar{Z}^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda, X)})}{\bar{Z}^\beta(\{1\}_{\bar{\mathcal{P}}(\Lambda)})} \tag{5.10}$$

We state the main result of this section.

Theorem 5.1. Let Φ be translational invariant or of finite range. Then for small complex β

$$\bar{\rho}^\beta(F) = \rho^\beta(F)$$

where the state ρ^β is obtained by the limit

$$\rho^\beta(F) = \lim_{\Lambda \uparrow Z^\nu} \rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})$$

as in Theorem 4.1(b).

We devote the rest of this section to the proof of the theorem.

Proof of Theorem 5.1. Because of Theorem 4.1(b), it suffices to show the theorem for real β. The basic strategy is to compare the expansion of $\rho^\beta(F)(\{1\}_{\mathcal{P}(\Lambda)})$ in Theorem 2.4 to that of $\bar{\rho}^\beta(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)})$ in (5.9). We show that the difference between the two expansions tends to zero as $\Lambda \to Z^\nu$.

We break up the sum $\sum_{\phi \subseteq X}$ in (5.9) into two terms:

$$\sum_{\substack{\phi \subseteq X: \\ X \cap \Lambda \neq \phi \\ X \cap X_0 \neq \phi}} = \sum_{\substack{\phi \subseteq X: \\ X \subseteq \Lambda \\ X \cap X_0 \neq \phi \ (X \neq \phi)}} + \sum_{\substack{\phi \subseteq X: \\ X \not\subseteq \Lambda,\, X \cap \Lambda \neq \phi \\ X \cap X_0 \neq \phi}}$$

$$= \sum_X{}^{(1)} + \sum_X{}^{(2)} \tag{5.11}$$

Let us denote for $j = 1, 2$

$$\bar{\rho}^\beta_{(j)}(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)}) = \sum_X{}^{(j)} F(X_0, X; \bar{f})\bar{g}(X \cup X_0) \tag{5.12}$$

We will show that

$$\bar{\rho}^\beta_{(2)}(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)}) \to 0 \quad \text{as } \Lambda \to Z^\nu \tag{5.13}$$

Assume that we can prove (5.13). Then

$$\begin{aligned}\bar{\rho}^\beta(F) &= \bar{\rho}^\beta(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)}) \\ &= \lim_{\Lambda \uparrow Z^\nu} \bar{\rho}^\beta_{(1)}(F)(\{1\}_{\bar{\mathcal{P}}(\Lambda)})\end{aligned} \tag{5.14}$$

as a consequence of (5.9) and (5.11)–(5.12). We then show that the right-hand side of (5.14) converges to $\rho^{\beta}(F)$ for small β.

We first prove (5.13). Following the method used in the proof of Lemma 3.1, we obtain the bounds

$$g_{\Lambda}(X) \leqslant e^{c|\beta||X|}$$
$$\left|\prod_{i=1}^{n} \delta^{X_i}\bar{f}(\{0\}_{\bar{\mathscr{P}}(\Lambda)})\right| \leqslant \|F\|\left(\prod_{i=1}^{n} |\beta|\,\|\Phi(X_i)\|\right)e^{c|\beta||\cup X_i|} \tag{5.15}$$

Using (5.15) and following the arguments used in the proof of Theorem 3.3 step by step, one may easily check that the right-hand side of (5.9) converges absolutely and uniformly in Λ. From this it follows that $\bar{\rho}^{\beta}_{(2)}(F)(\{1\}_{\bar{\mathscr{P}}(\Lambda)})$ tends to zero as $\Lambda \to Z^{\nu}$ for small real β. This completes the proof of (5.13).

We now prove that

$$\rho^{\beta}(F)(\{1\}_{\mathscr{P}(\Lambda)}) = \lim_{\Lambda \uparrow Z^{\nu}} \bar{\rho}^{\beta}_{(1)}(F)(\{1\}_{\bar{\mathscr{P}}(\Lambda)}) \tag{5.16}$$

By the Araki's Gibbs condition in (5.4) we have the identity

$$\prod_{i=1}^{n} \delta^{X_i}\bar{f}(\{0\}_{\bar{\mathscr{P}}(\Lambda)}) = \prod_{i=1}^{n} \delta^{X_i} f(\{0\}_{\mathscr{P}(\Lambda)}) \qquad \text{for} \quad X \subset \Lambda, \quad X = \cup X_i \tag{5.17}$$

where $f(\{0\}_{\mathscr{P}(\Lambda)})$ is given by (2.3). We use (5.17) and (5.12) to get

$$\bar{\rho}^{\beta}_{(1)}(F)(\{1\}_{\bar{\mathscr{P}}(\Lambda)}) = \sum_{\substack{\phi \subseteq X: \\ X \subset \Lambda \\ X \cap X_0 \neq \phi\ (X \neq \phi)}} K^{\beta}(X_0, X; f)\bar{g}_{\Lambda}(X \cup X_0)$$

The above expression converges absolutely and uniformly in Λ by (5.15) and Proposition 3.2 for small β. Thus by comparing (5.17) to the expansion of $\rho^{\beta}(F)(\{1\}_{\mathscr{P}(\Lambda)})$ in Theorem 2.4 term by term, (5.16) will be proved if we can show that

$$\lim_{\Lambda \uparrow Z^{\nu}} g_{\Lambda}(X) = \lim_{\Lambda \uparrow Z^{\nu}} \bar{g}_{\Lambda}(X) \tag{5.18}$$

for small β.

To prove (5.18) we again apply the method of integral equations for $\bar{g}_{\Lambda}(X)$. Using the method used in (4.5)–(4.8) one may get the following equation: For a fixed $x_0 \in X$

$$\bar{g}_{\Lambda}(X) = \bar{g}_{\Lambda}(X - x_0) - \sum_{\substack{\phi \neq S \in \bar{\mathscr{P}}(\Lambda, X - x_0) \\ x_0 \in S}} K^{\beta}(\{x_0\}, S; \bar{Z}^{\beta})\bar{g}_{\Lambda}(X \cup S) \tag{5.19}$$

Again we break up $\sum_S$ into two terms:

$$\sum_{\substack{\phi \neq S \in \overline{\mathcal{P}}(\Lambda, X - x_0) \\ x_0 \in S}} = \sum_{\substack{S \in \mathcal{P}(\Lambda \setminus X - x_0) \\ x_0 \in S}} + \sum_{\substack{S: \\ S \not\subset \Lambda \setminus X - x_0 \\ x_0 \in S}} \tag{5.20}$$

We define

$$\epsilon_\Lambda(X) = \sum_{\substack{S: \\ S \not\subset \Lambda \setminus X - x_0 \\ x_0 \in S}} K^\beta\big(\{x_0\}, S; \overline{Z}^\beta\big)\bar{g}_\Lambda(X \cup S) \tag{5.21}$$

Since the right-hand side of (5.19) converges absolutely and uniformly in Λ by (5.15) and the argument used in the proof of Proposition 4.2(a), we conclude from (5.19)–(5.21) that

$$\epsilon_\Lambda(X) \to 0 \quad \text{as} \quad \Lambda \to Z^\nu \tag{5.22}$$

for small β. We also note that

$$\prod_{i=1}^{n} \delta^{S_i}\overline{Z}^\beta\big(\{0\}_{\overline{\mathcal{P}}(\Lambda)}\big) = \prod_{i=1}^{n} \delta^{S_i} Z^\beta\big(\{0\}_{\mathcal{P}(S)}\big) \qquad \text{for} \quad S \subset \Lambda, \quad S = \cup S_i$$

by (5.4) and (5.6). By the argument used in Corollary 4.3 we obtain the equation

$$\bar{g}_\Lambda = \mathbb{1} + \chi_\Lambda K \chi_\Lambda \bar{g}_\Lambda + \epsilon_\Lambda \tag{5.23}$$

where K is defined by (4.9). Since $\epsilon_\Lambda(X) \to 0$ as $\Lambda \to Z^\nu$ by (5.22) and since the limit

$$g = \lim_{\Lambda \uparrow Z^\nu} g_\Lambda$$

satisfies the equation

$$g = \mathbb{1} + Kg \tag{5.24}$$

by the result in Section 4, the limit

$$\bar{g} = \lim_{\Lambda \uparrow Z^\nu} \bar{g}_\Lambda$$

exists and satisfies the equation

$$\bar{g} = 1 + K\bar{g} \tag{5.25}$$

As a consequence of (5.24) and (5.25) we have proved (5.18). This completes the proof of Theorem 5.1. ■

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